Solution to Final Examination

MAT1322D, Fall 2016

BFBCCBCD

Part I. Multiple-choice Questions ($3 \times 8 = 24$ marks)

1. Let R be the region under the graph of $y = x^3 + x$ and above the x-axis in interval [0, 1]. Solid B is obtained by revolving R **about the y-axis**. Then the volume of B is

(A) $\frac{15}{8}\pi$; (B) $\frac{16}{15}\pi$; (C) $\frac{5}{4}\pi$; (D) $\frac{16}{3}\pi$; (E) $\frac{8}{15}\pi$; (F) $\frac{15}{6}\pi$.

Answer. (B) Use the cylindrical shell method, the area is

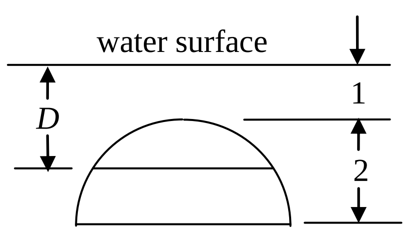
$$A = 2\pi\int_0^1 x(x^3+x)dx = 2\pi\left[\frac{1}{5}x^5+\frac{1}{3}x^3\right]_{x=0}^{1} = \frac{16}{15}\pi.$$

2. Suppose a surface of the shape of the upper half of a disk with radius 2 meters is submerged into water (with density ρ kg/m^3) so that the top of the half disk is 1 meters under the water. (See the figure below). Let D be the depth of a horizontal stripe of the surface. Let g be the acceleration of gravity. Then the force acting on this surface is calculate by the definite integral

(A) $\rho g\int_1^3 D\sqrt{2^2-(3-D)^2}\,dD$; (B) $2\rho g\int_0^2 (D+1)\sqrt{2^2-(3-D)^2}\,dD$;

(C) $\rho g\int_0^2 D\sqrt{2^2-(1+D)^2}\,dD$; (D) $2\rho g\int_1^3 (D+1)\sqrt{2^2-D^2}\,dD$;

(E) $2\rho g\int_1^3 D\sqrt{2^2-(1+D)^2}\,dD$; (F) $2\rho g\int_1^3 D\sqrt{2^2-(3-D)^2}\,dD$.



Solution. (F) Consider a stripe of height dD at depth D. The area of this stripe is

$A(D) = 2\sqrt{2^2-(3-D)^2}\,dD$, and the pressure is $P(D) = \rho g D$.

The force acting on this stripe is $F(D) = A(D)P(D) = 2\rho g D\sqrt{2^2-(3-D)^2}\,dD$.

The total force is $F = \int_1^3 F(D)dD = 2\rho g\int_1^3 D\sqrt{2^2-(3-D)^2}dD$.

3. Consider improper integral $\int_0^1 \frac{1}{2\sqrt{x}-x}dx$. Which one of the following argument is true?

(A) When $0 < x < 1$, $\frac{1}{2\sqrt{x}-x} < \frac{1}{\sqrt{x}}$. Since $\int_0^1 \frac{1}{\sqrt{x}}dx$ diverges, $\int_0^1 \frac{1}{2\sqrt{x}-x}dx$ diverges.

(B) When $0 < x < 1$, $\frac{1}{2\sqrt{x}-x} < \frac{1}{\sqrt{x}}$. Since $\int_0^1 \frac{1}{\sqrt{x}}dx$ converges, $\int_0^1 \frac{1}{2\sqrt{x}-x}dx$ converges.

(C) When $0 < x < 1$, $\frac{1}{2\sqrt{x}-x} > \frac{1}{2\sqrt{x}}$. Since $\int_0^1 \frac{1}{2\sqrt{x}}dx = \frac{1}{2}\int_0^1 \frac{1}{\sqrt{x}}dx$ converges, $\int_0^1 \frac{1}{2\sqrt{x}-x}dx$ converges.

(D) When $0 < x < 1$, $\frac{1}{2\sqrt{x}-x} > \frac{1}{2\sqrt{x}}$. Since $\int_0^1 \frac{1}{2\sqrt{x}}dx = \frac{1}{2}\int_0^1 \frac{1}{\sqrt{x}}dx$ diverges, $\int_0^1 \frac{1}{2\sqrt{x}-x}dx$ diverges.

(E) When $0 < x < 1$, $\frac{1}{2\sqrt{x}-x} > \frac{1}{x}$. Since $\int_0^1 \frac{1}{x}dx$ diverges, $\int_0^1 \frac{1}{2\sqrt{x}-x}dx$ diverges.

(F) When $0 < x < 1$, $\frac{1}{2\sqrt{x}-x} < \frac{1}{x}$. Since $\int_0^1 \frac{1}{x}dx$ converges, $\int_0^1 \frac{1}{2\sqrt{x}-x}dx$ converges.

Solution. (B) When x is close to 0, x is much smaller than $\sqrt{x}$. The behaviour of integral $\int_0^1 \frac{1}{2\sqrt{x}-x}dx$ is similar to $\int_0^1 \frac{1}{2\sqrt{x}}dx = \frac{1}{2}\int_0^1 \frac{1}{\sqrt{x}}dx$. Since $\int_0^1 \frac{1}{\sqrt{x}}dx$ converges, we guess that this integral converges. Then (A), (D) and (E) are false. (C) is false because the convergence of the integral of a small function does not imply the convergence of the integral of a bigger function. (F) is false because integral $\int_0^1 \frac{1}{x}dx$ diverges.

4. Suppose Euler's method with step size $h = 0.1$ is used to find an approximation of $y(0.2)$, where $y(t)$ is the solution to the initial-value problem $y' = 2t - y$, $y(0) = 1$. Which one of the following is closest to the answer?

(A) 0.79; (B) 0.81; (C) 0.83; (D) 0.85; (E) 0.87; (F) 0.89.

Solution. (C)

i	t_i	y_i
0	0	1
1	0.1	$1 + 0.1 \times (2 \times 0 - 1) = 0.9$

2	0.2	$0.9 + 0.1 \times (2 \times 0.1 - 0.9) = 0.83$

5. If $y = f(t)$ is the solution of the initial-value problem $y' = y^2 - 2y$, $y(0) = 1$. Which one of the following values is closest to the value $y(1)$?

(A) 0.22; (B) 0.23; (C) 0.24; (D) 0.25; (E) 0.26; (F) 0.27.

Solution. (C) Separating variables, $\int \frac{dy}{y(y-2)} = \int dt$. Hence,

$\int \frac{dy}{y(y-2)} = \frac{1}{2}\int \left(\frac{1}{y-2} - \frac{1}{y}\right)dy = \frac{1}{2}\ln\left|\frac{y-2}{y}\right| = t + C$. Then $\frac{y-2}{y} = Ke^{2t}$, where $K = \pm e^{2C} \neq 0$.

By the initial condition, $K = -1$. $y = \frac{2}{1+e^{2t}}$. $y(1) = \frac{2}{1+e^{2}} \approx 0.24$.

6. Suppose the Maclaurin series of a function $y = f(x)$ is given by

$$f(x) = \frac{1}{3-2} - \frac{1}{3^2-2^2}x + \frac{1}{3^3-2^3}x^2 - \ldots = \sum_{n=0}^{\infty}(-1)^n \frac{1}{3^{n+1}-2^{n+1}}x^n .$$

Then the fifth derivative of $f(x)$ at $x = 0$ is $f^{(5)}(0) =$

(A) $\frac{1}{665}$; (B) $-\frac{24}{133}$; (C) $-\frac{1}{665}$; (D) $\frac{24}{133}$; (E) $\frac{24}{665}$; (F) $-\frac{24}{665}$.

Solution. (B) In the Maclaurin series, the coefficient of x^5 is $\frac{f^{(5)}(0)}{5!} = (-1)^5\frac{1}{3^6-2^6} = -\frac{1}{665}$.

Then $f^{(5)}(0) = -\frac{120}{665} = -\frac{24}{133}$.

7. If a function $z = f(x, y)$ is defined implicitly by the equation $xyz - z^3 + 2x^2y^2 = 4$, then the directional derivative of this function in the direction of vector $\mathbf{u} = (-3, 4)$ at the point $(3, -1, 2)$ is

(A) $-\frac{2}{3}$; (B) $\frac{2}{5}$; (C) -2; (D) -10; (E) $\frac{16}{5}$; (F) 2.

Solution. (C) Let $F(x, y, z) = xyz - z^3 + 2x^2y^2 - 4$. $F_x = yz + 4xy^2$, $F_y = xz + 4x^2y$, $F_z = xy + 3z^2$.

At point $(3, -1, 2)$, $F_x = 10$, $F_y = -30$, $F_z = -15$. Then $z_x = -\frac{F_x}{F_z} = \frac{2}{3}$, $z_y = -\frac{F_y}{F_z} = -2$.

The unit vector in the direction of $\mathbf{u}$ is $\mathbf{v} = \frac{\mathbf{u}}{|\mathbf{u}|} = \left(-\frac{3}{5}, \frac{4}{5}\right)$. The directional derivative is

$D_{\mathbf{u}}(3, -1, 2) = -\frac{3}{5}\left(\frac{2}{3}\right) + (-2)\,\frac{4}{5} = -\frac{2}{5} - \frac{8}{5} = -2\,.$

8. The equation of the tangent plane of the graph of the function $z = 2xy - y^2$ at the point $x = 1$ and $y = 2$ is

(A) $z = 2x + 4y$; (B) $z = 4x + 2y$; (C) $z = 2x + 2y$;
(D) $x = 4x - 2y$; (E) $z = 2x - 4y$; (F) $z = 4x - 4y$.

Solution. (D) $z_x = 2y$, $z_y = 2x - 2y$. When $x = 1$ and $y = 2$, $z_x(1, 2) = 4$, $z_y(1, 2) = -2$, $z(1, 2) = 0$. The equation of the tangent plane is $z = 4(x - 1) - 2(y - 2)$, or $z = 4x - 2y$.

Part II. Long Answer Questions (26 marks)

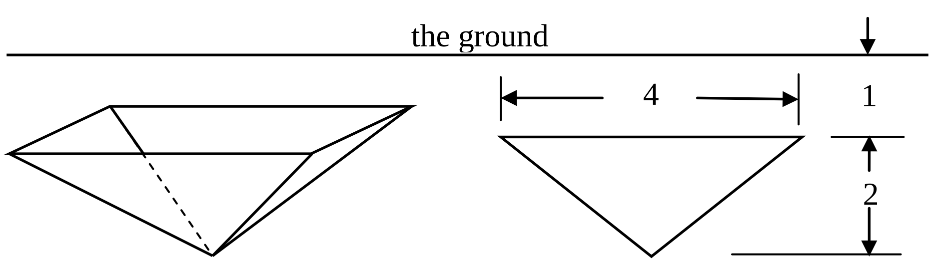
1. (4 marks) Use **the definition of the improper integral** to find the value of $\int_0^4 \frac{1}{\sqrt{4-x}}\,dx$.

Solution.

$$\int_0^4 \frac{1}{\sqrt{4-x}}\,dx = \lim_{b\to 4^-}\int_0^b \frac{1}{\sqrt{4-x}}dx = \lim_{b\to 4^-}\int_{4-b}^4 \frac{1}{\sqrt{u}}du = 2\lim_{b\to 4^-}\left[\sqrt{u}\,\right]_{u=4-b}^4 = 2\lim_{b\to 4^-}(2-\sqrt{4-b}) = 4\,.$$

To get the full mark of this question: You must use the definition of improper integral. Change the limits in definite integral when variable substitution is used. Correctly use the notation and symbols. Do not miss dx in the integral. Use the limit sign in each step.

2. (4 marks) Suppose a container is of the shape of an inverted rectangular pyramid as shown in the following figure is buried so that the top is 1 meter under the ground. A vertical cross section of the container is show on the right. The top of the container is a square of side-length 4 meters, and the depth of the container is 2 meters. Suppose the container is filled with oil of density ρ = 800 kg/m^3. (Assume the acceleration of gravity is $g \approx 9.81$ m/sec^2.)



Find the work, in Joules, needed to pump the oil in the container to the ground.

Solution. Consider a horizontal layer of oil with thickness dx at x meters above the bottom of the container.

The area of this cross section is $A(x) = (2x)^2$.
The volume of this layer is $V(x) = A(x)dx = 4x^2\, dx$.
The weight of this layer is $w(x) = \rho g V(x) = 4\rho g x^2 dx$.
This layer has to be pumped for a distance $3 - x$.
The work needed to pump this layer of oil to the ground is $W(x) = w(x)(3 - x) = 4\rho g x^2(3 - x)\, dx$.

The total work needed is $W = \int_0^2 W(x)dx = 4\rho g\int_0^2 x^2(3-x)dx \approx 1.25 \times 10^5$ Joule.

Alternative solutions:

a. If you let x be the distance between a horizontal layer of oil and the top of the container, then the integral will be $4\rho g\int_0^2 (2-x)^2(x+1)dx$.

b. If you let x be the distance between a horizontal layer of oil and the ground level, then the integral will be $4\rho g\int_1^3 (3-x)^2 x dx$.

3. (4 marks) Consider the series $\sum_{n=1}^{\infty}\frac{(-1)^n}{2^n\sqrt{2n-1}}(x+1)^n$. Find the value(s) of x where this series is absolutely convergent, conditionally convergent, or divergent.

Solution. The center of the series is -1. $c_n = \frac{(-1)^n}{2^n\sqrt{2n-1}}$. The radius of convergence is $R = \lim_{n\to\infty}\left|\frac{c_n}{c_{n+1}}\right| = \lim_{n\to\infty}\frac{2^{n+1}\sqrt{2(n+1)-1}}{2^n\sqrt{2n-1}} = \lim_{n\to\infty}\frac{2^{n+1}\sqrt{2n+1}}{2^n\sqrt{2n-1}} = 2$. Then this series is absolutely convergent when $-1 - 2 < x < -1 + 2$, i.e., $-3 < x < 1$, and it diverges when $x < -3$ or $x > 1$. When $x = -3$, the series become $\sum_{n=1}^{\infty}\frac{(-1)^n}{2^n\sqrt{2n-1}}(-2)^n = \sum_{n=1}^{\infty}\frac{1}{\sqrt{2n-1}}$. Since $\frac{1}{\sqrt{2n-1}} > \frac{1}{\sqrt{2n}}$, and $\sum_{n=1}^{\infty}\frac{1}{\sqrt{2n}} = \frac{1}{\sqrt{2}}\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ diverges, this series diverges when $x = -3$. When $x = 1$, this series become $\sum_{n=1}^{\infty}\frac{(-1)^n}{2^n\sqrt{2n-1}}2^n = \sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{2n-1}}$, which converges by the alternating series test. Since $\sum_{n=1}^{\infty}\frac{1}{\sqrt{2n-1}}$ diverges, this series is conditionally convergent at $x = 1$.

In summary, this series absolutely converges when $-3 < x < 1$, it conditionally converges when $x = 1$, and it diverges when $x \le -3$ or $x > 1$.

4. (6 marks) Use an appropriate test method to determine whether each of the following series is convergent or divergent.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{\sqrt{4n^5 - n - 1}}$; (b) $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{3^n - 2^n}{3^{n+1} - 2^{n+1}}$.

Solution. (a) Since this is a positive series, we can use the limit comparison test.

Let $a_n = \dfrac{n}{\sqrt{4n^5 - n - 1}}$ and $b_n = \dfrac{1}{n\sqrt{n}}$. Then

$\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n}{\sqrt{4n^5 - n - 1}} n\sqrt{n} = \lim_{n\to\infty} \sqrt{\frac{n^5}{4n^5 - n - 1}} = \frac{1}{2}$. Since the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}}$, as a p-series with $p = 3 / 2$, converges, this series converges.

(b) Since $\displaystyle\lim_{n\to\infty} \frac{3^n - 2^n}{3^{n+1} - 2^{n+1}} = \lim_{n\to\infty} \frac{(3^n - 2^n)/3^n}{(3^{n+1} - 2^{n+1})/3^n} = \lim_{n\to\infty} \frac{1-(2/3)^n}{3-2(2/3)^n} = \frac{1}{3}$, the general term of this series does not approach 0. This series diverges.

5. (4 marks) The Maclaurin series of the function $y = \ln(1 + x)$ is

$$\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}.$$

Find the first four non-zero terms of the Maclaurin series of the function $F(x) = \int_0^x \ln(1+t^2)dt$.

Solution. Use substitution, $x = t^2$. Then $\ln(1 + t^2) = t^2 - \dfrac{t^4}{2} + \dfrac{t^6}{3} - \dfrac{t^8}{4} + \ldots = \displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{t^{2n}}{n}$.

$$\int_0^x \ln(1+t^2)dt = \int_0^x \left(t^2 - \frac{t^4}{2} + \frac{t^6}{3} - \frac{t^8}{4} + \ldots \right) dt = \left[\frac{1}{3}t^3 - \frac{1}{10}t^5 + \frac{1}{21}t^7 - \frac{1}{36}t^9 + \ldots \right]_{t=0}^{x}$$
$$= \frac{1}{3}x^3 - \frac{1}{10}x^5 + \frac{1}{21}x^7 - \frac{1}{36}x^9 + \ldots .$$

6. Consider function $f(x, y) = \dfrac{1}{\sqrt{x^2 + y^2 - 1}}$.

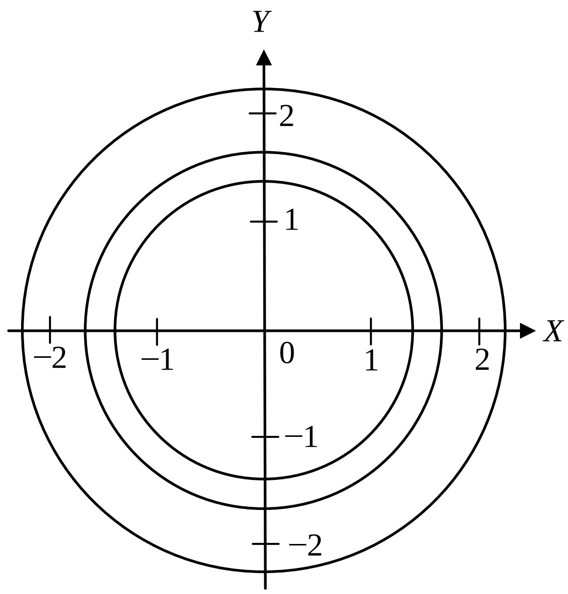
(a) Find the domain of this function.

Answer. The domain is $x^2 + y^2 > 1$, i.e., the region outside the unit circle.

(b) Draw level curves of this function when $f(x, y) = \frac{1}{2}, \frac{3}{4}, 1$.

Solution.

$\sqrt{x^2+y^2-1} = 2$, $x^2 + y^2 = 5$, a circle centered at the origin with radius $\sqrt{5} \approx 2.24$.
$\sqrt{x^2+y^2-1} = \frac{4}{3}$, $x^2 + y^2 = 1 + \frac{16}{9} = \frac{25}{9}$, a circle centered at the origin with radius $\frac{5}{3} \approx 1.67$.
$\sqrt{x^2+y^2-1} = 1$, $x^2 + y^2 = 2$, a circle centered at the origin with radius $\sqrt{2} \approx 1.41$.



(c) In the direction of which vector does this function increases the fastest at the point $(x, y) = (2, -1)$?

$f_x = -\frac{x}{(x^2+y^2-1)^{3/2}}$, $f_y = -\frac{y}{(x^2+y^2-1)^{3/2}}$.

When $x = 2$, $y = -1$, $f_x(2, -1) = -\frac{1}{4}$, $f_y(2, -1) = \frac{1}{8}$. This function increases the fastest in the direction of the vector $\mathbf{u} = \left(\frac{-1}{4}, \frac{1}{8}\right)$. Any multiple of this vector works, say $\mathbf{u} = (-2, 1)$.